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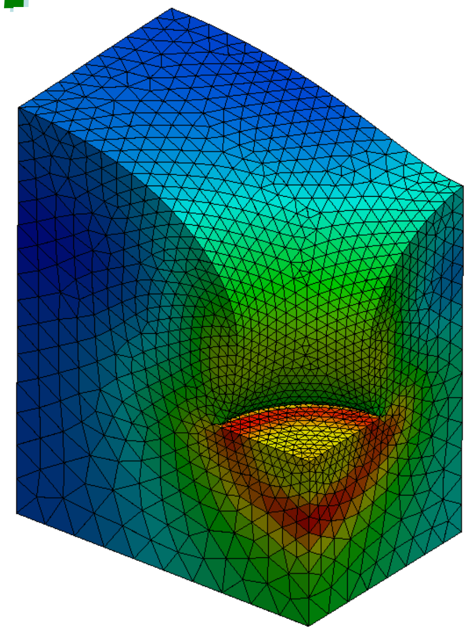
# A Stable Rezoning Method for Large Deformation Finite Element Analysis using Incremental Equilibrium Equation

This talk indirectly relates to Particle/Meshfree methods.
I beg your patience till the end.



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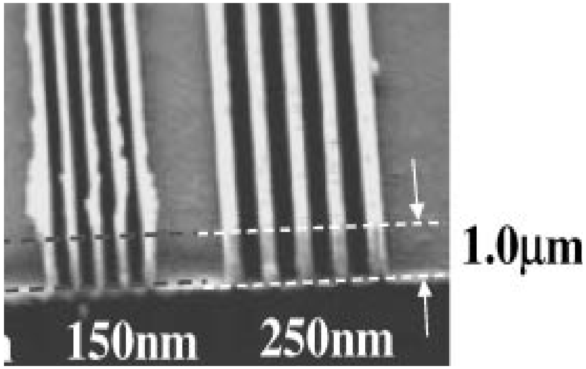
# Motivation and Background

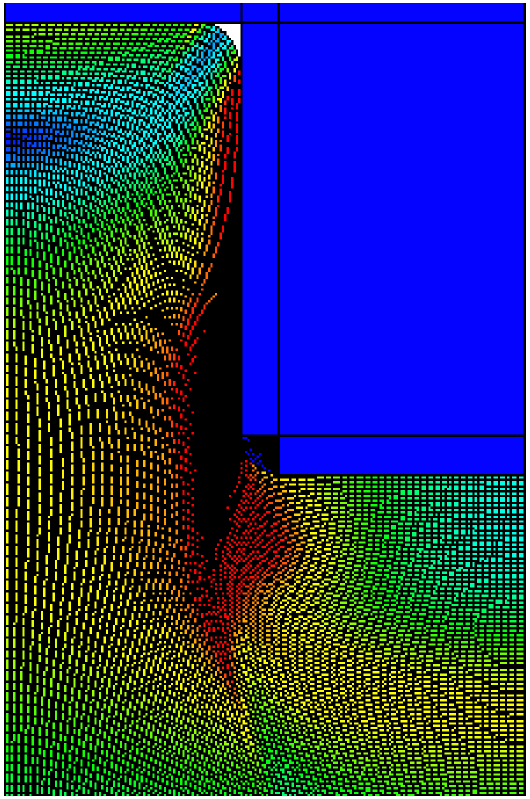
## Motivation

We want to solve **severely large deformation** problems **accurately and stably**!
(Final target: thermal nanoimprinting)




## Background

Finite elements are **distorted** in a short time, thereby resulting in convergence failure.



FE rezoning method (*h*-adaptive mesh-to-mesh solution mapping) is indispensable.

# Methods for Forming Simulation

| | Software | Accuracy | Stability |
|---|---|---|---|
| One Step Method | HyperForm<br>FASTFORM | ★ | ★★★★★ |
| Dynamic-Explicit FE Rezoning | LS-DYNA<br>PAM-STAMP | ★★ | ★★★ |
| Static-Explicit FE Rezoning | ASU/P-form | ★★★ | ★★ |
| Static-*Implicit* FE Rezoning | ABAQUS<br>MARC | ★★★★★ | ★ |

Most of the rezoning researches try to improve this.

Our approach tries to improve this with a new idea.

# Objective

Develop an **accurate and stable**
***implicit*** FE rezoning method
for large deformation problems
with a new idea.

New Idea: adopting implicit FE formulation based on the incremental equilibrium equation (IEE)

***Table of Body Contents***

① Derivation of the IEE for static-***implicit*** analysis
② Formulation of our ***implicit*** FE rezoning method based on the IEE
③ Verification analysis in 2D
④ Demonstration analysis in 3D

# ①
# Derivation of the incremental equilibrium equation (IEE) for static-***implicit*** analysis

# Virtual Work Equation in Rate Form

$$\int_{\Omega(t)} \dot{\mathbf{\Pi}}_t^T(t) : \delta \boldsymbol{F}_t(t) \, \mathrm{d}\Omega$$

Work Conjugate

The static-explicit FE formulation uses the same Eq.

$$= \int_{\Gamma(t)} \dot{\underline{\boldsymbol{t}}}_t(t) \cdot \delta \boldsymbol{u} \, \mathrm{d}\Gamma + \int_{\Omega(t)} \rho \dot{\boldsymbol{g}} \cdot \delta \boldsymbol{u} \, \mathrm{d}\Omega$$

$\square_t$: Variable in the Current Configuration,

$\delta\square$: Variation, $\dot{\square}$: Material Time Derivative,

$\mathbf{\Pi}$: 1st Piola-Kirchhoff Stress Tensor,

$\boldsymbol{F}$: Deformation Gradient Tensor,

$\underline{\boldsymbol{t}}$: Surface Traction Vector,

$\Omega$: Analysis Domain, $\Gamma$: Domain Boundary,

$\boldsymbol{u}$: Displacement vector, $\rho$: Density,

$\boldsymbol{g}$: Body Force Vector

# Linearization and Discretization

$$\int_{\Omega(t)} \dot{\mathbf{\Pi}}_t^T(t) : \delta \boldsymbol{F}_t(t)\ \mathrm{d}\Omega$$

Work Conjugate

$$= \int_{\Gamma(t)} \dot{\underline{\boldsymbol{t}}}_t(t) \cdot \delta \boldsymbol{u}\ \mathrm{d}\Gamma + \int_{\Omega(t)} \rho \dot{\boldsymbol{g}} \cdot \delta \boldsymbol{u}\ \mathrm{d}\Omega$$

Linearization in Time

$$\dot{\mathbf{\Pi}}_t^T(t) \simeq \Delta \mathbf{\Pi}_t^T / \Delta t, \quad \dot{\underline{\boldsymbol{t}}}_t(t) \simeq \Delta \underline{\boldsymbol{t}}_t / \Delta t, \quad \dot{\boldsymbol{g}} \simeq \Delta \boldsymbol{g} / \Delta t$$

Galerkin Discretization

$$\delta \boldsymbol{F}_t(t) \simeq [B_{\mathrm{N}}]\{\delta u\}, \quad \delta \boldsymbol{u} \simeq \{N\}\{\delta u\}$$

***Fully Implicit*** Time Advancing

$$\sum_{e \in \mathbb{E}} \int_{\Omega_e^+} [B_{\mathrm{N}}^+]^T \{\Delta \Pi_t^T\}\ \mathrm{d}\Omega$$

$$= \sum_{s \in \mathbb{S}} \int_{\Gamma_s^+} [N^+]^T \{\Delta \underline{t}_t\}\ \mathrm{d}\Gamma + \sum_{e \in \mathbb{E}} \int_{\Omega_e^+} \rho^+ [N^+]^T \{\Delta g\}\ \mathrm{d}\Omega$$

$\square^+$:Trial Variable,
$\mathbb{E}$ : Set of Elements,
$\mathbb{S}$ : Set of Element Faces

# Incremental Equilibrium Equation (IEE)

$$\sum_{e\in\mathbb{E}}\int_{\Omega_e^+}[B_{\mathrm{N}}^+]^T\{\Delta\Pi_t^T\}\,\mathrm{d}\Omega = \sum_{s\in\mathbb{S}}\int_{\Gamma_s^+}[N^+]^T\{\Delta\underline{t}_t\}\,\mathrm{d}\Gamma + \sum_{e\in\mathbb{E}}\int_{\Omega_e^+}\rho^+[N^+]^T\{\Delta g\}\,\mathrm{d}\Omega$$

$\square^+$:Trial Variable,
$\mathbb{E}$ : Set of Elements,
$\mathbb{S}$ : Set of Element Faces

Let the left-hand side be $\{\Delta f^{\mathrm{int}}\}$,
the right-hand side be $\{\Delta f^{\mathrm{ext}}\}$.

The implicit IEE: $\{\Delta f^{\mathrm{ext}}\} - \{\Delta f^{\mathrm{int}}\} = \{0\}$

Avoid error accumulation through timesteps.

The implicit IEE (secondary form):

$$(\{f^{\mathrm{ext}}\} + \{\Delta f^{\mathrm{ext}}\}) - (\{f^{\mathrm{int}}\} + \{\Delta f^{\mathrm{int}}\}) = \{0\}$$

We use the secondary form in the actual implementation.

# Comparison of IEE to Standard EE

[Standard Static-Implicit EE]

$$\{f^{\mathrm{ext}}\} - \{f^{\mathrm{int}}\} = \{0\},$$

$$\{f^{\mathrm{ext}}\} = \sum_{s\in\mathbb{S}} \int_{\Gamma_s^+} [N^+]^T \{\underline{t}^+\}\,\mathrm{d}\Gamma + \sum_{e\in\mathbb{E}} \int_{\Omega_e^+} \rho^+ [N^+]^T \{g\}\,\mathrm{d}\Omega,$$

$$\{f^{\mathrm{int}}\} = \sum_{e\in\mathbb{E}} \int_{\Omega_e^+} [B_{\mathrm{L}}^+]^T \{T^+\}\,\mathrm{d}\Omega,$$

Cauchy Stress

[Implicit IEE]

$$\{\Delta f^{\mathrm{ext}}\} - \{\Delta f^{\mathrm{int}}\} = \{0\},$$

$$\{\Delta f^{\mathrm{ext}}\} = \sum_{s\in\mathbb{S}} \int_{\Gamma_s^+} [N^+]^T \{\Delta \underline{t}_t\}\,\mathrm{d}\Gamma + \sum_{e\in\mathbb{E}} \int_{\Omega_e^+} \rho^+ [N^+]^T \{\Delta g\}\,\mathrm{d}\Omega,$$

$$\{\Delta f^{\mathrm{int}}\} = \sum_{e\in\mathbb{E}} \int_{\Omega_e^+} [B_{\mathrm{N}}^+]^T \{\Delta \Pi_t^T\}\,\mathrm{d}\Omega,$$

1st P-K Stress increment

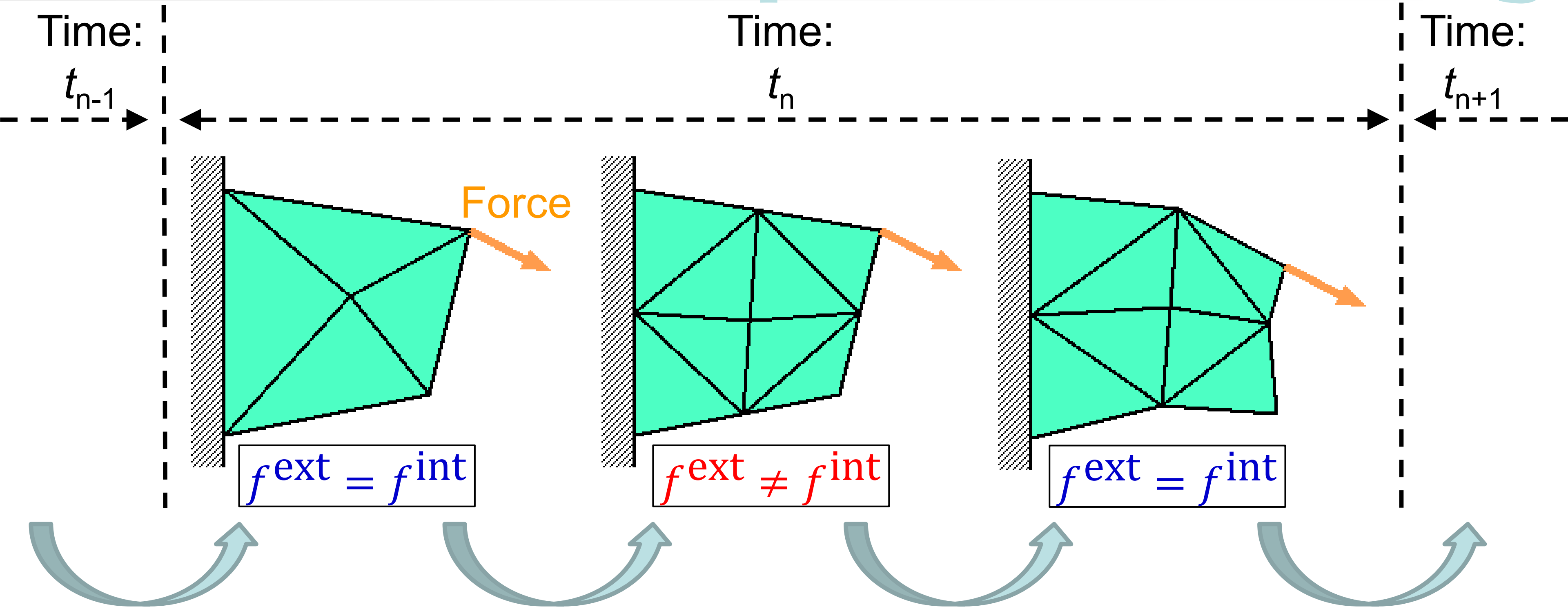
②

Formulation of

our ***implicit*** FE rezoning method

based on the IEE

# Conventional Implicit FE Rezoning



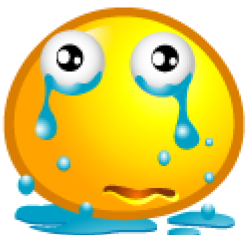

Solve
Standard EE

Remesh
&
Map States

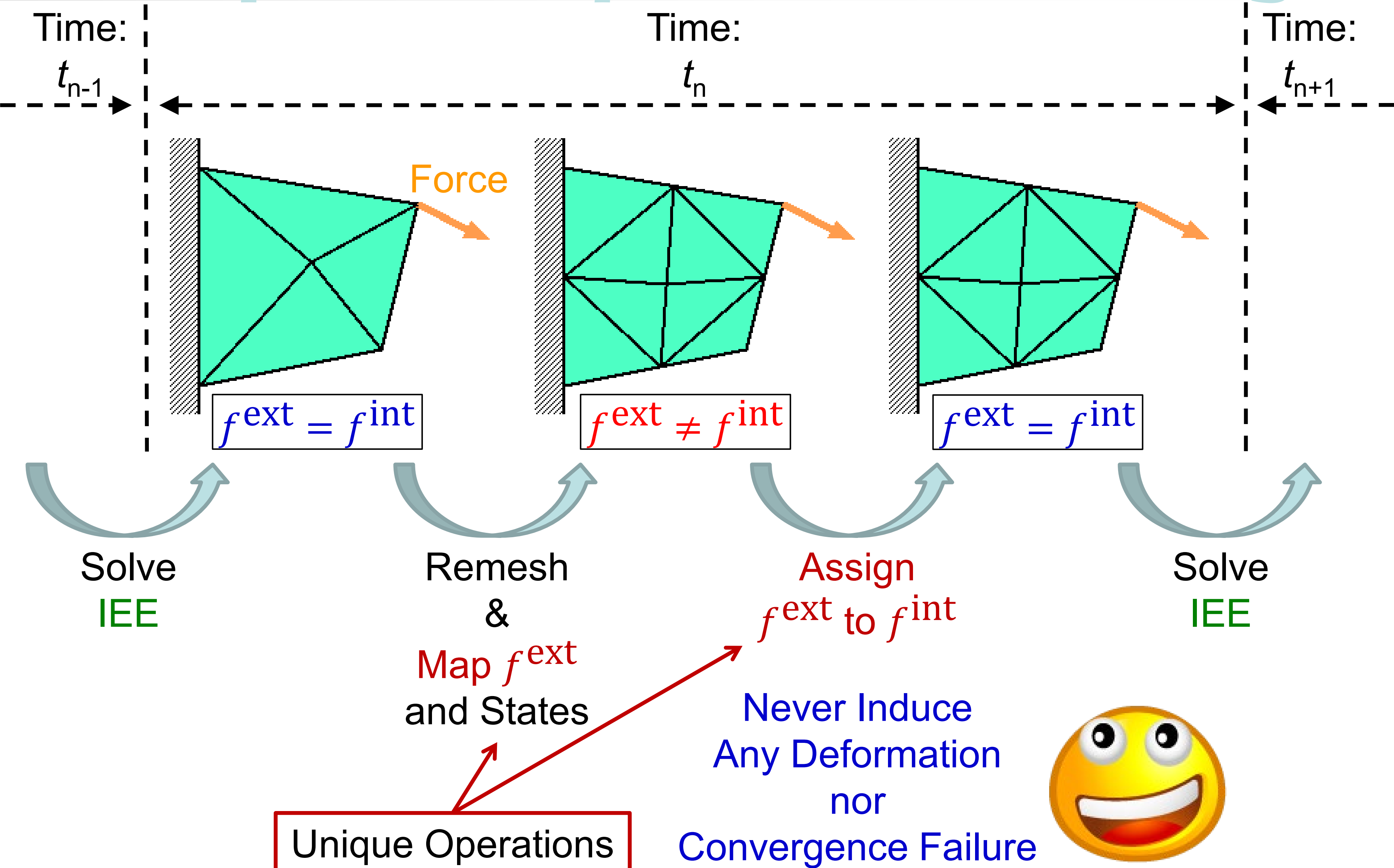
**Resolve**
Standard EE

Solve
Standard EE

Induce
Large Deformation
and
Convergence Failure



# Proposed Implicit FE Rezoning



# Flowchart of the Proposed Method

- Start of timestep loop
  - Assume initial $\{\Delta u\}$
  - Start of (implicit) Newton-Raphson loop
    - Calculate trial states
    - Calculate $\{\Delta f^{\text{ext}}\}, \{\Delta f^{\text{int}}\}$, and $[K]$
    - Convergence check
    - Solve $[K]\{\delta u\} = (\{f^{\text{ext}}\} + \{\Delta f^{\text{ext}}\}) - (\{f^{\text{int}}\} + \{\Delta f^{\text{int}}\})$
    - Substitute $\{\Delta u\} + \{\delta u\}$ for $\{\Delta u\}$
  - Substitute $\{f^{\text{ext}}\} + \{\Delta f^{\text{ext}}\}$ for $\{f^{\text{ext}}\}$
  - Substitute $\{f^{\text{int}}\} + \{\Delta f^{\text{int}}\}$ for $\{f^{\text{int}}\}$
  - Update States
  - Rezone if necessary

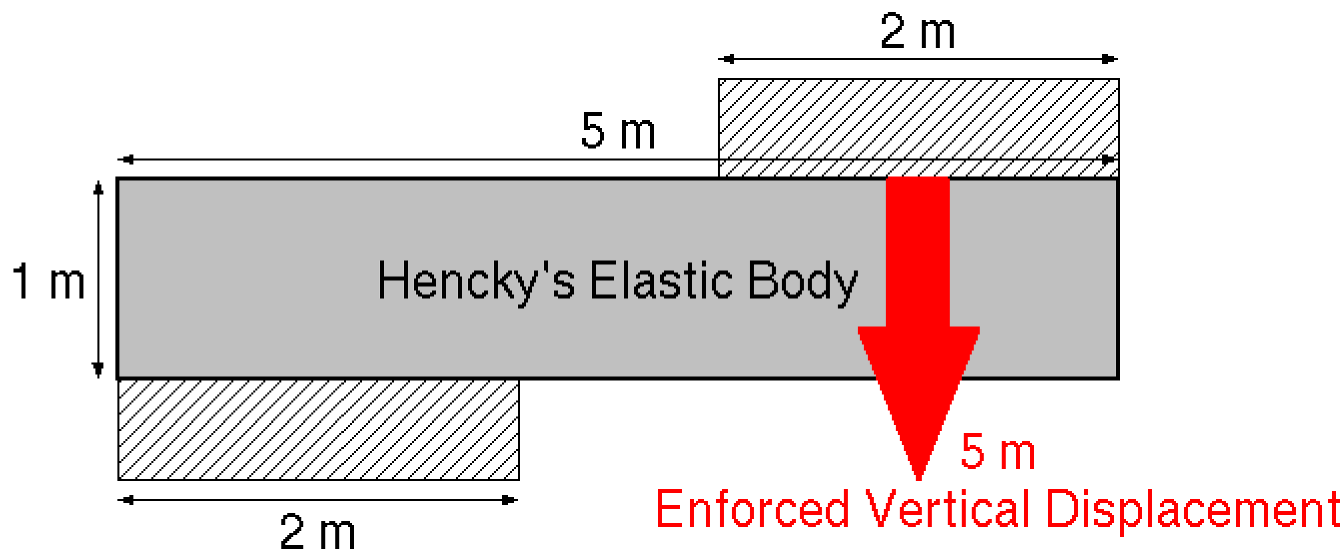
Almost the same as the conventional implicit method except the green parts

# Proposed vs. Conventional

| | Proposed *Implicit* FE Rezoning | Conventional *Implicit* FE Rezoning |
|---|---|---|
| Equation to be Solved | IEE | Standard EE |
| Mapping of $f^{\mathrm{ext}}$ | Required... | Unnecessary! |
| Equilibrium after Mapping | YES! | NO... |
| Unique Deformed Shape at a Time | YES! | NO... |
| Convergence Failure in Rezoning Process | NO! | YES... |

# ③ Verification Analysis in 2D

# Shearing of 2D Bar




- Static, 2D Plane-strain condition
- All 1st order triangular elements
- Global rezoning every 10 timesteps (much more frequent than necessary)
- remeshing with ANSYS GAMBIT

# Shearing of 2D Bar

- Material: Hencky's elastic body
  - constitutive equation in total strain form:

$$\boldsymbol{T} = \boldsymbol{C}_{\mathrm{L}} : \boldsymbol{E}$$

Cauchy Stress ∝ Hencky Strain

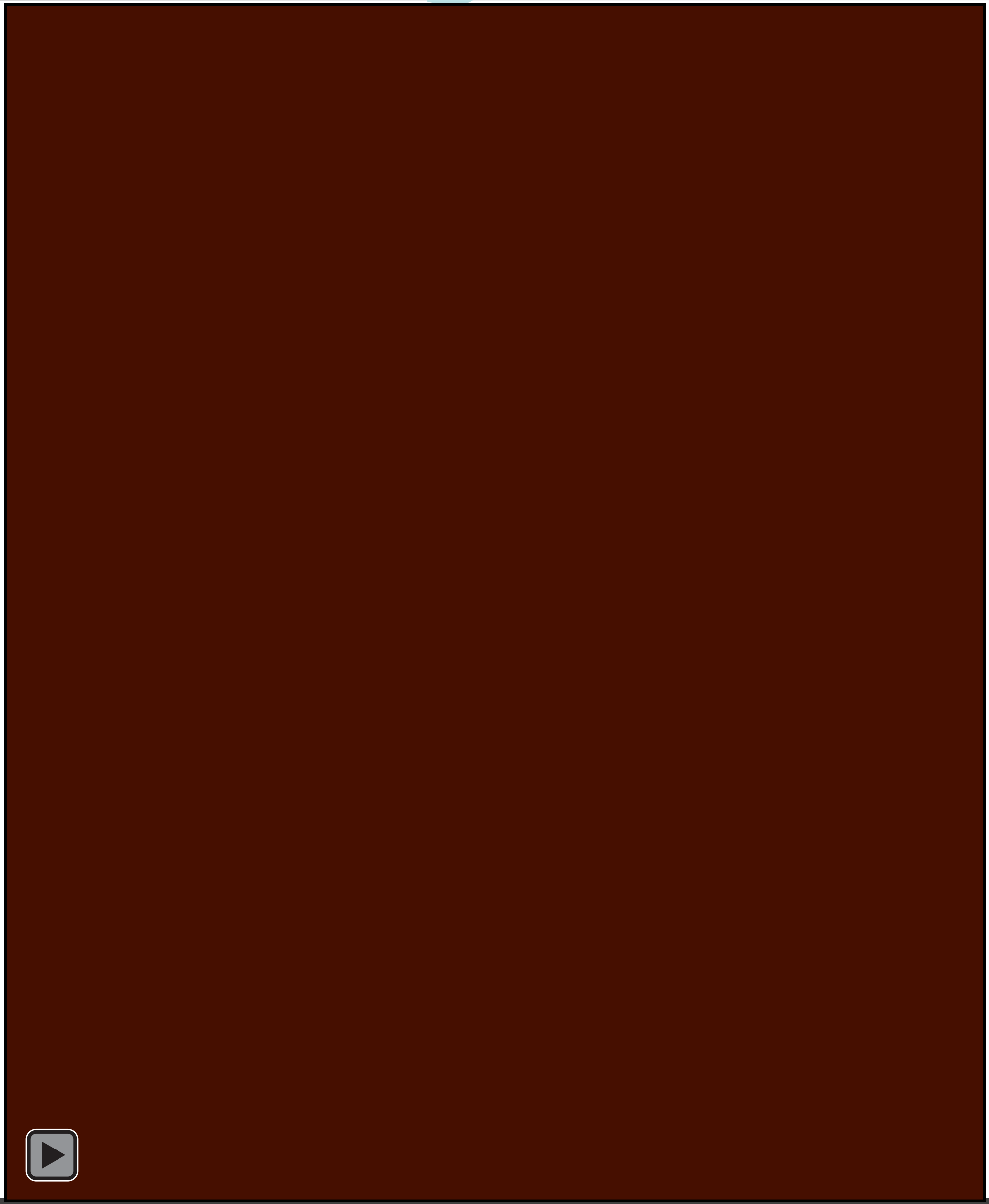
  - constitutive equation in rate form:

$$\overset{\circ}{\boldsymbol{T}} = \boldsymbol{C}_{\mathrm{L}} : \boldsymbol{D}$$

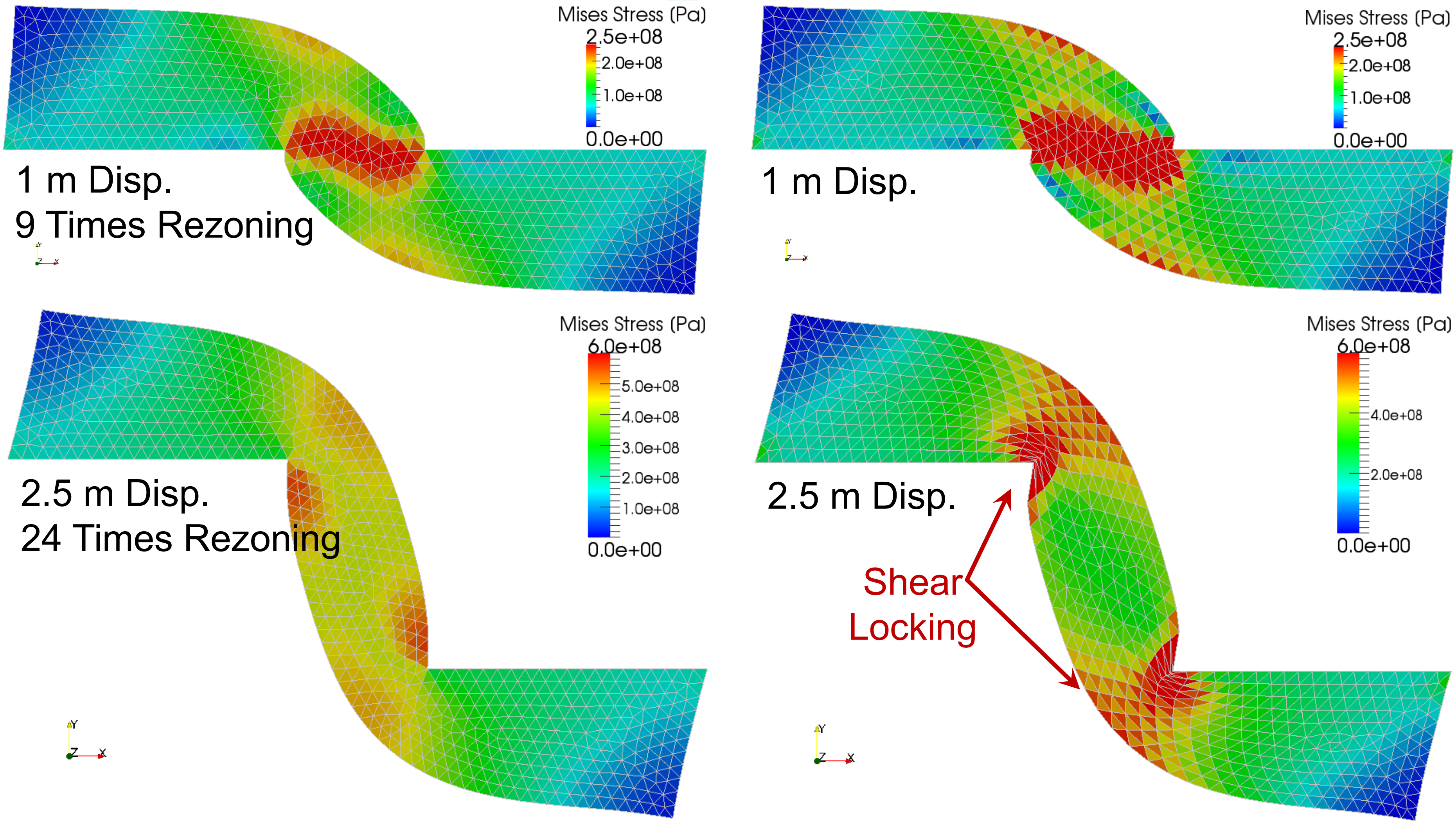
Jaumann Rate of Cauchy Stress ∝ Stretching

  - Young's modulus: 1 GPa; Poisson's Ratio: 0.3

# Shearing of 2D Bar



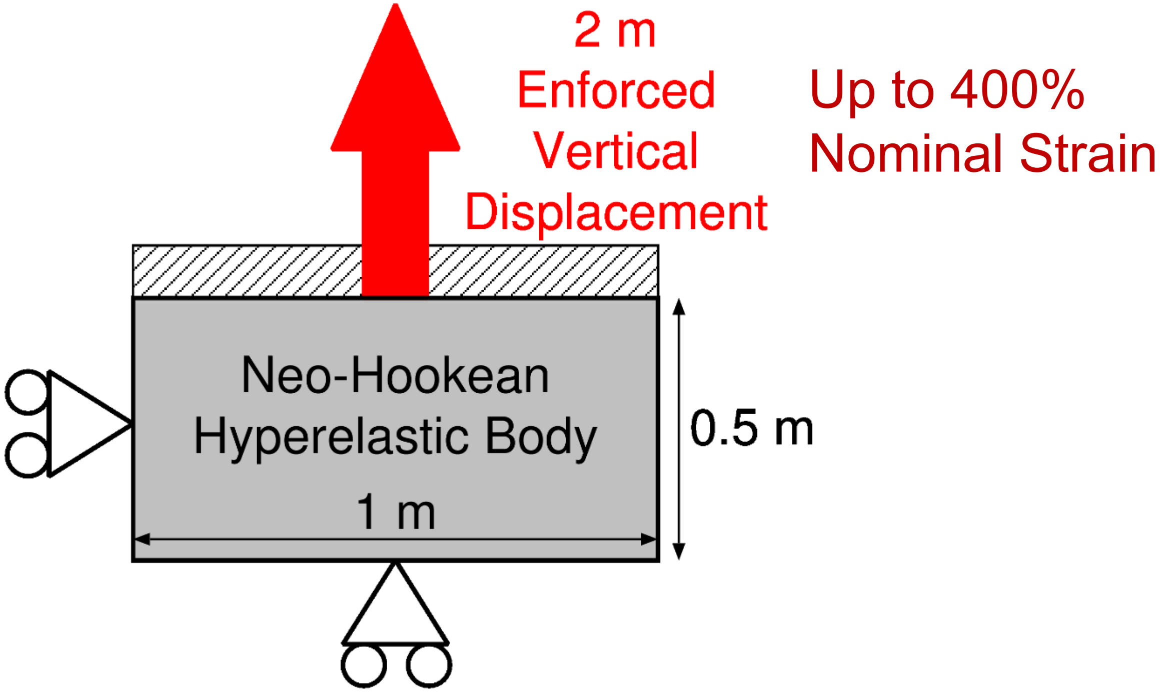
# Shearing of 2D Bar




Proposed Method

Standard Implicit FEM without Rezoning

# Tension of 2D Brick




- Static, 2D Plane-strain condition
- All 1st order triangular elements
- Global rezoning every 5 timesteps

# Tension of 2D Brick

■ Material: Neo-Hookean hyperelastic body

- Strain energy density function:

$$U = C_{10}(\bar{I}_1 - 3) + \frac{1}{D_1}(J-1)^2$$

- Constitutive equation in total strain form:

$$\boldsymbol{T} = \frac{2}{J}C_{10}\mathrm{dev}(\bar{\boldsymbol{B}}) + \frac{2}{D_1}(J-1)\boldsymbol{I}$$

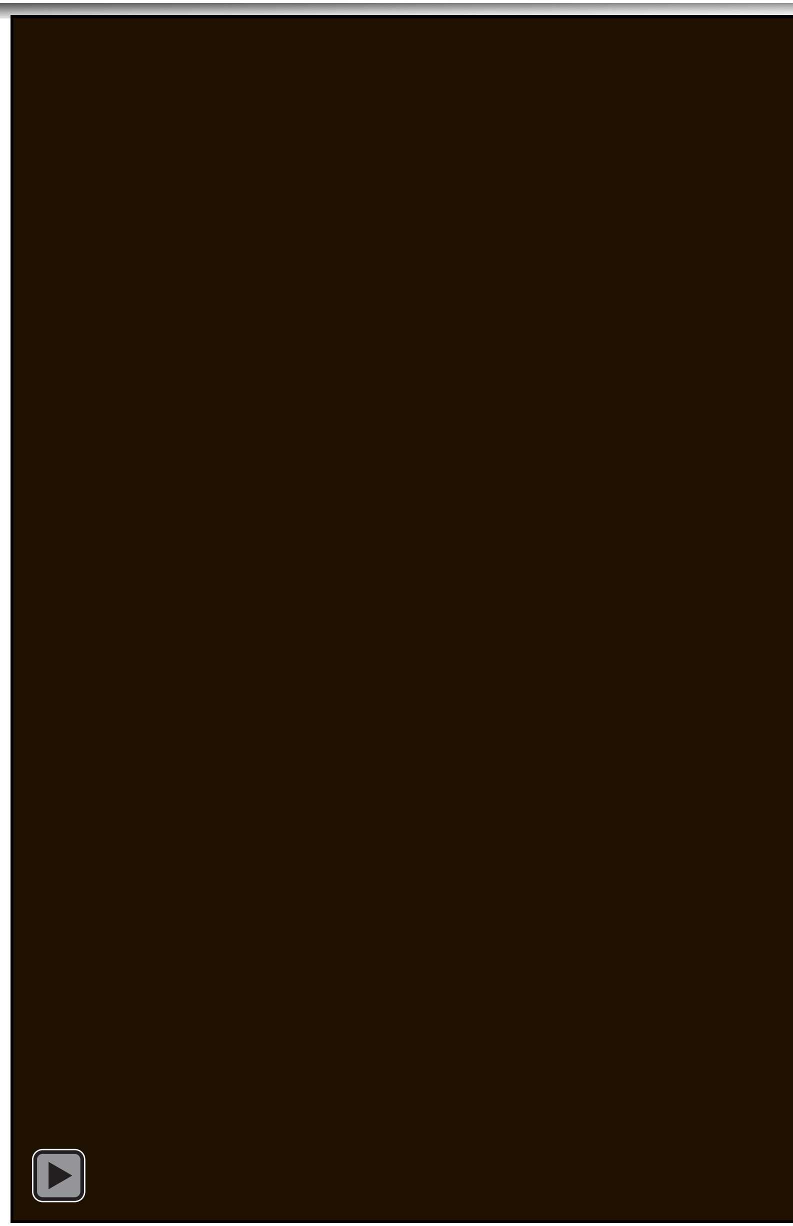
- Constitutive equation in strain rate form:

$$\overset{\circ}{\boldsymbol{T}} = \boldsymbol{C}_{\mathrm{L}}(\boldsymbol{F}) : \boldsymbol{D}$$

where $C_L(F)$ is obtained through a long hand calculation.

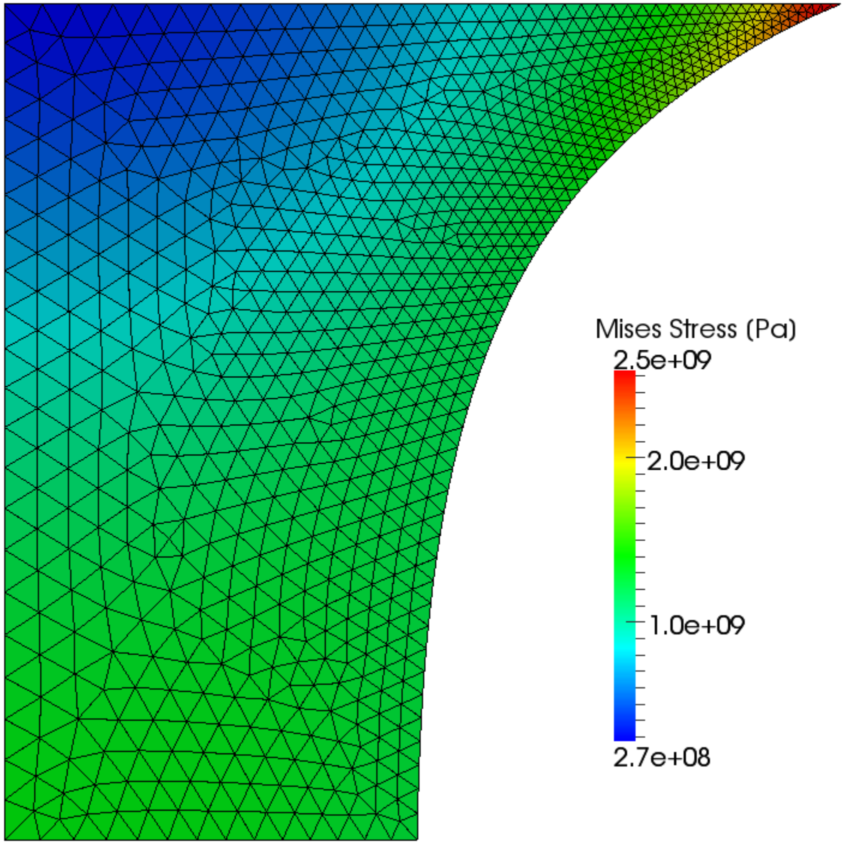
- $C_{10}$=0.172 GPa; $D_1$=0.6 GPa$^{-1}$

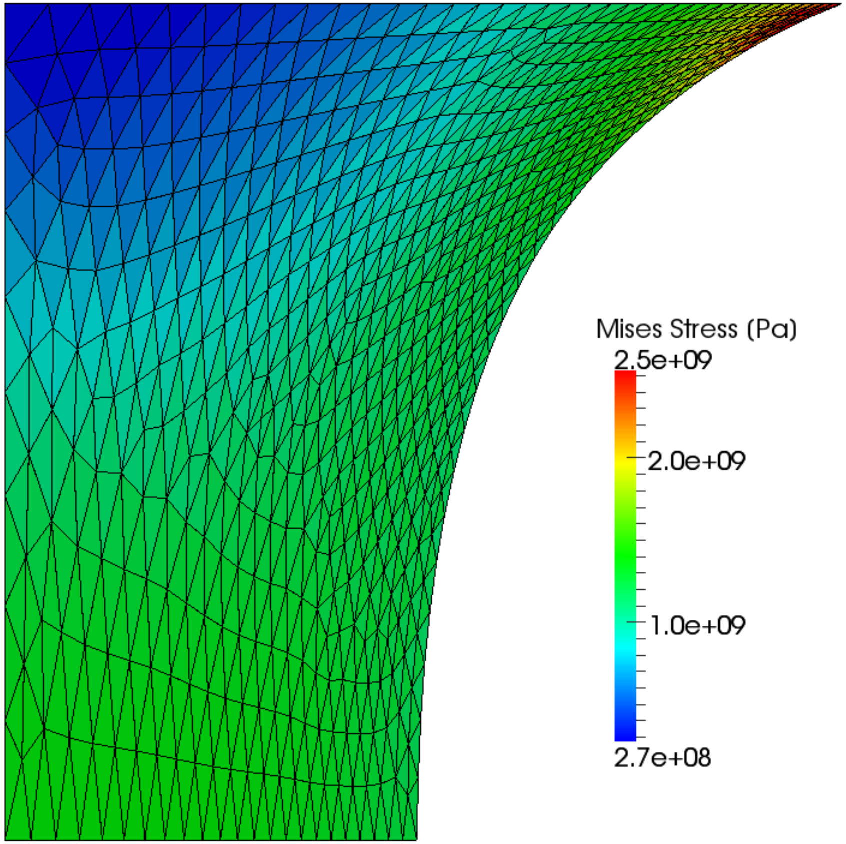
# Tension of 2D Brick



# Tension of 2D Brick

Proposed Method (19 Times Rezoning)

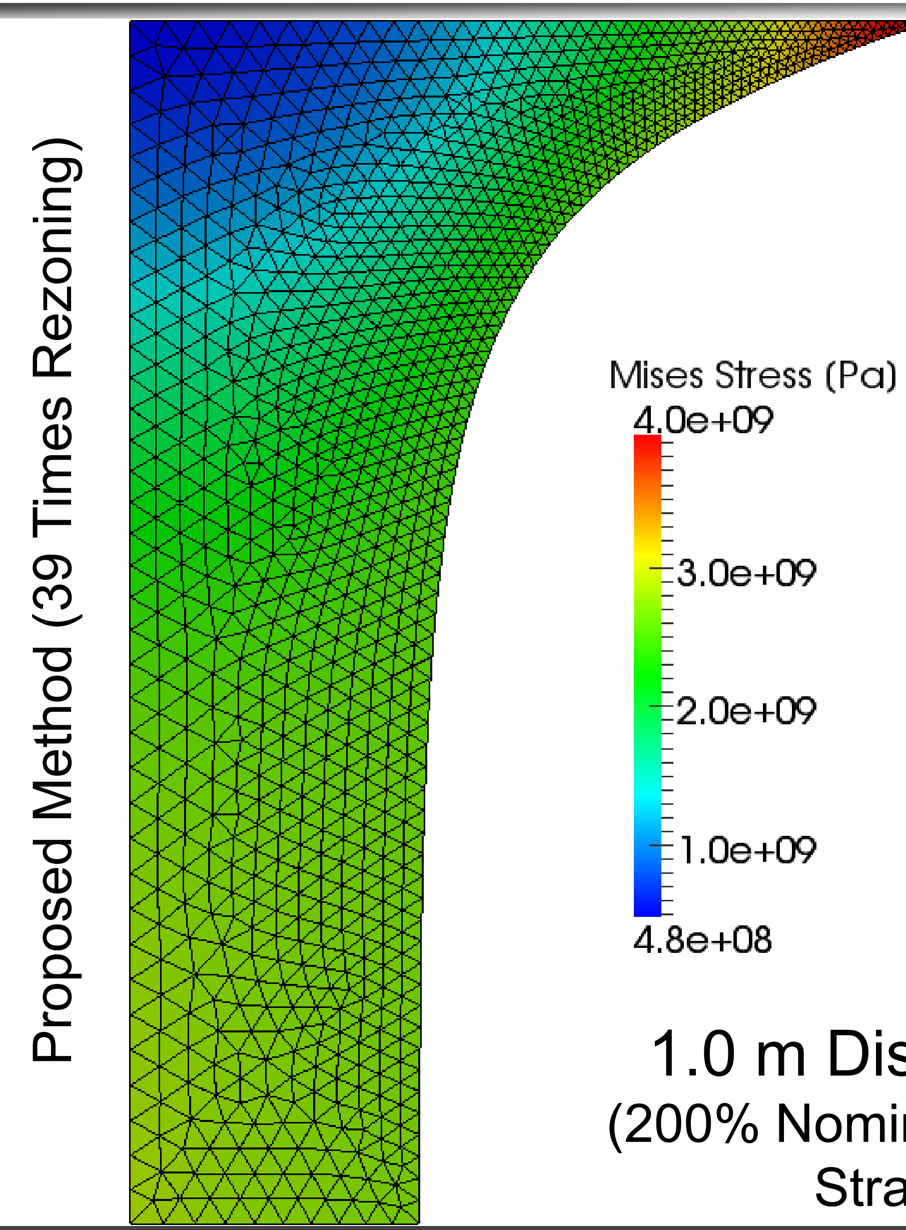


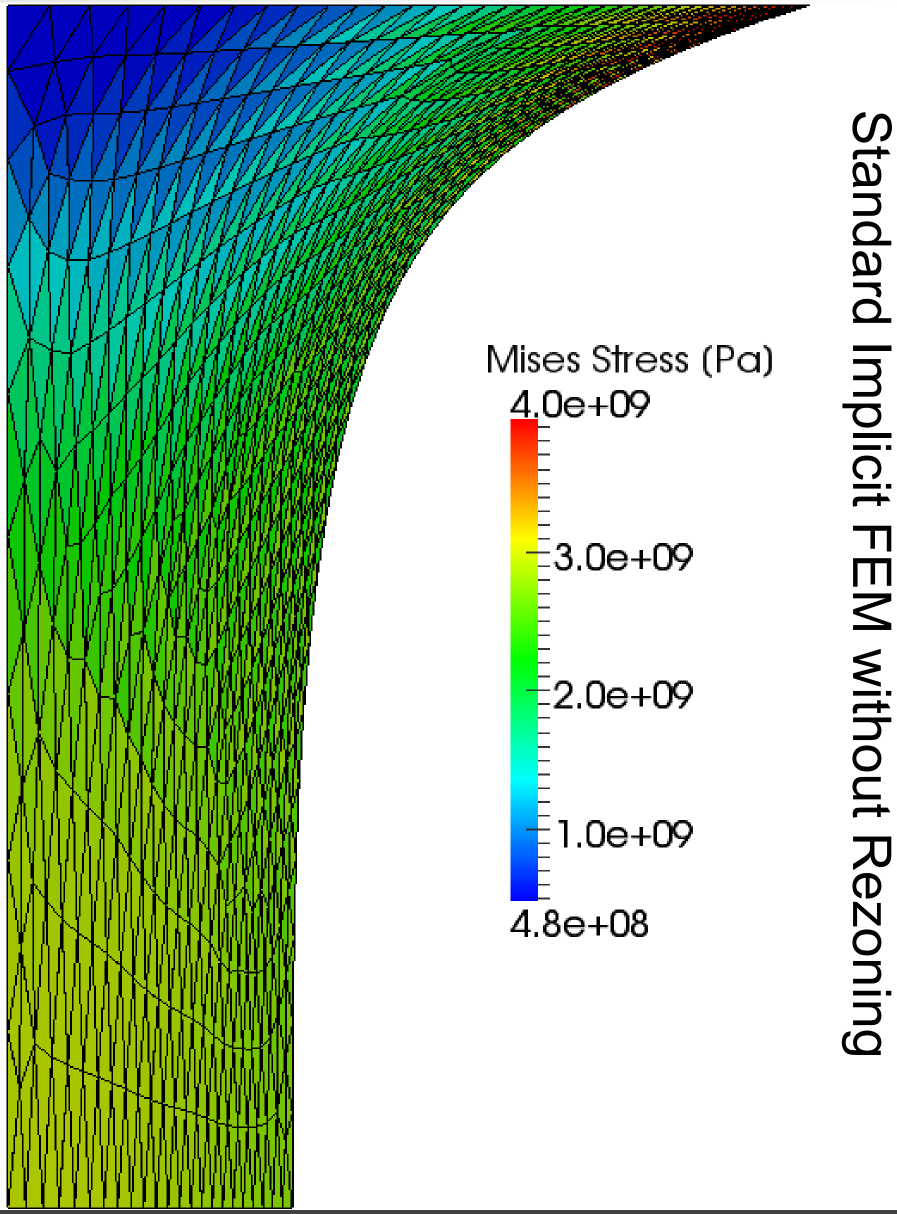

Standard Implicit FEM without Rezoning

0.5 m Disp.
(100% Nominal Strain)

# Tension of 2D Brick

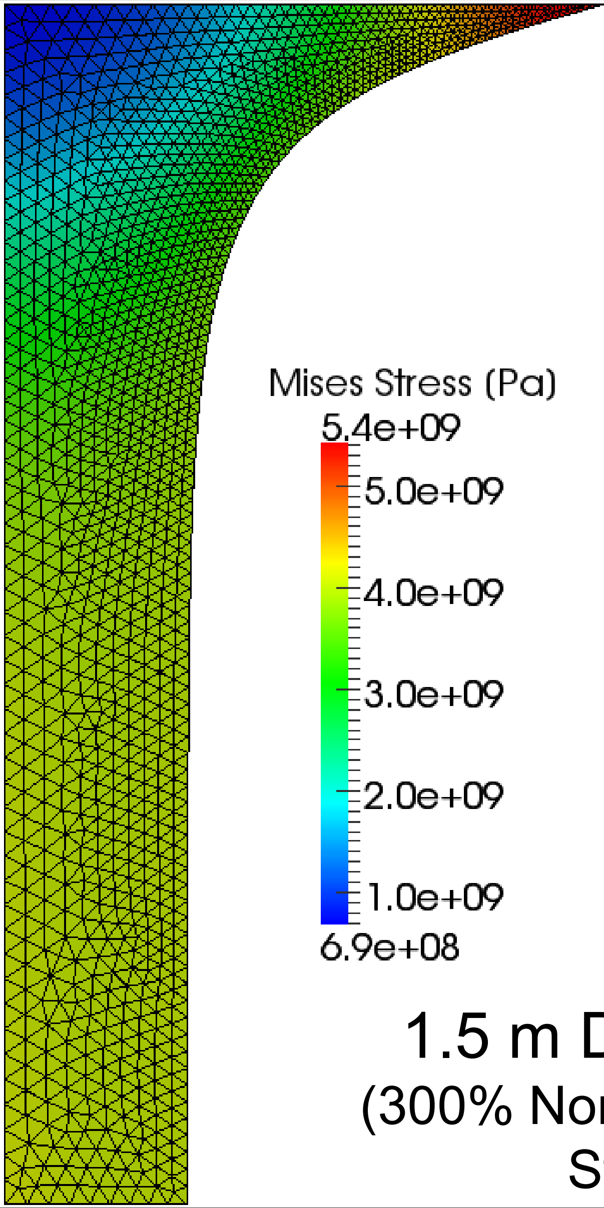



1.0 m Disp.
(200% Nominal Strain)

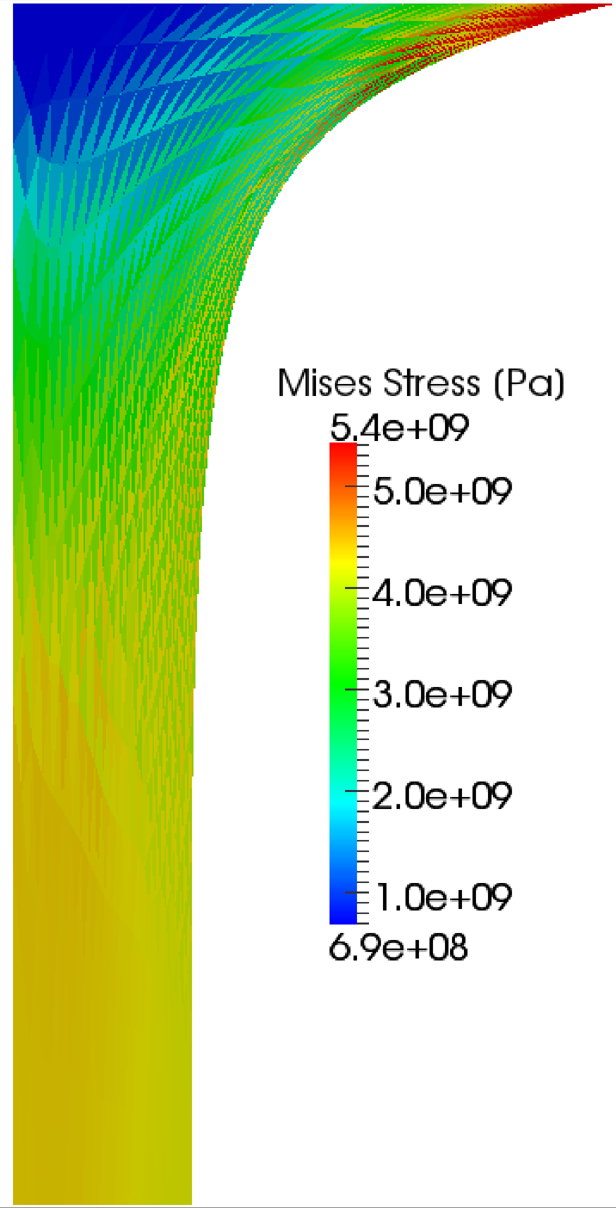


# Tension of 2D Brick

Proposed Method (59 Times Rezoning)



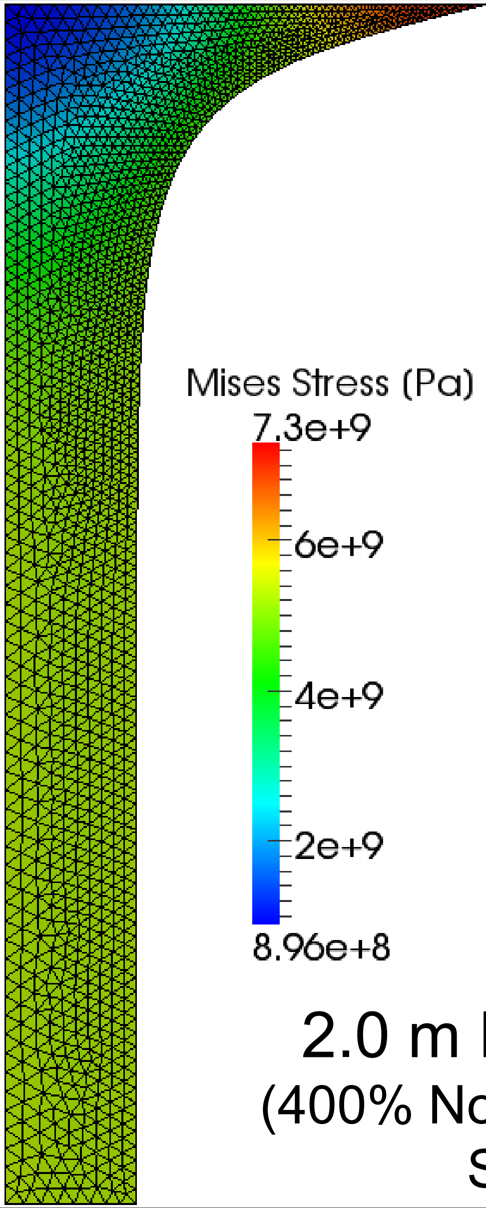

1.5 m Disp.
(300% Nominal Strain)



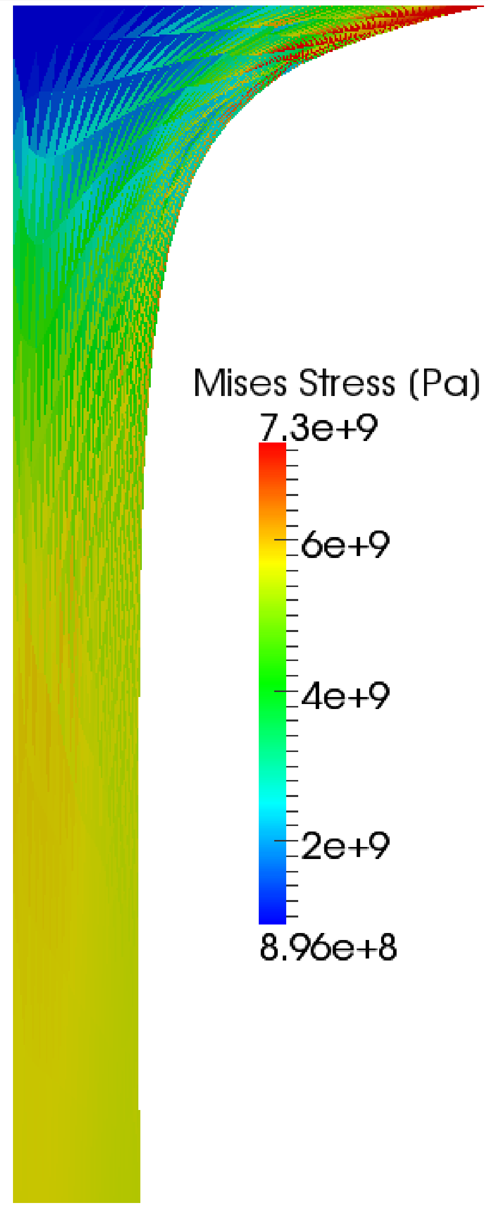

Standard Implicit FEM without Rezoning

# Tension of 2D Brick

Proposed Method (79 Times Rezoning)



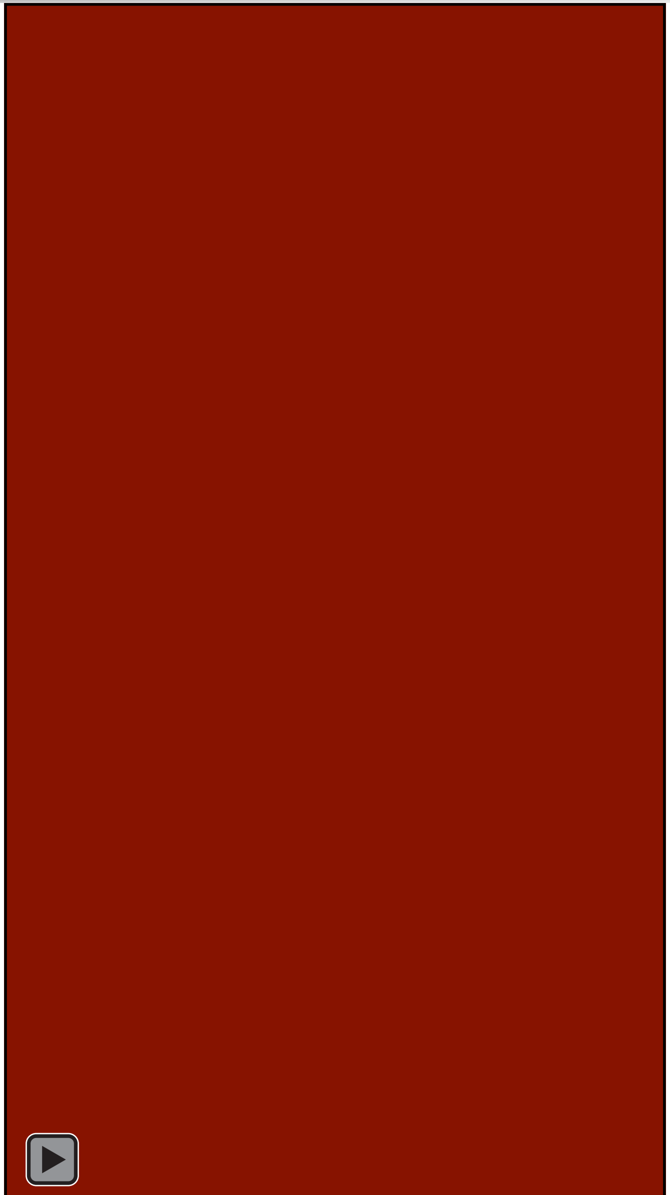

2.0 m Disp.
(400% Nominal Strain)




Standard Implicit FEM without Rezoning

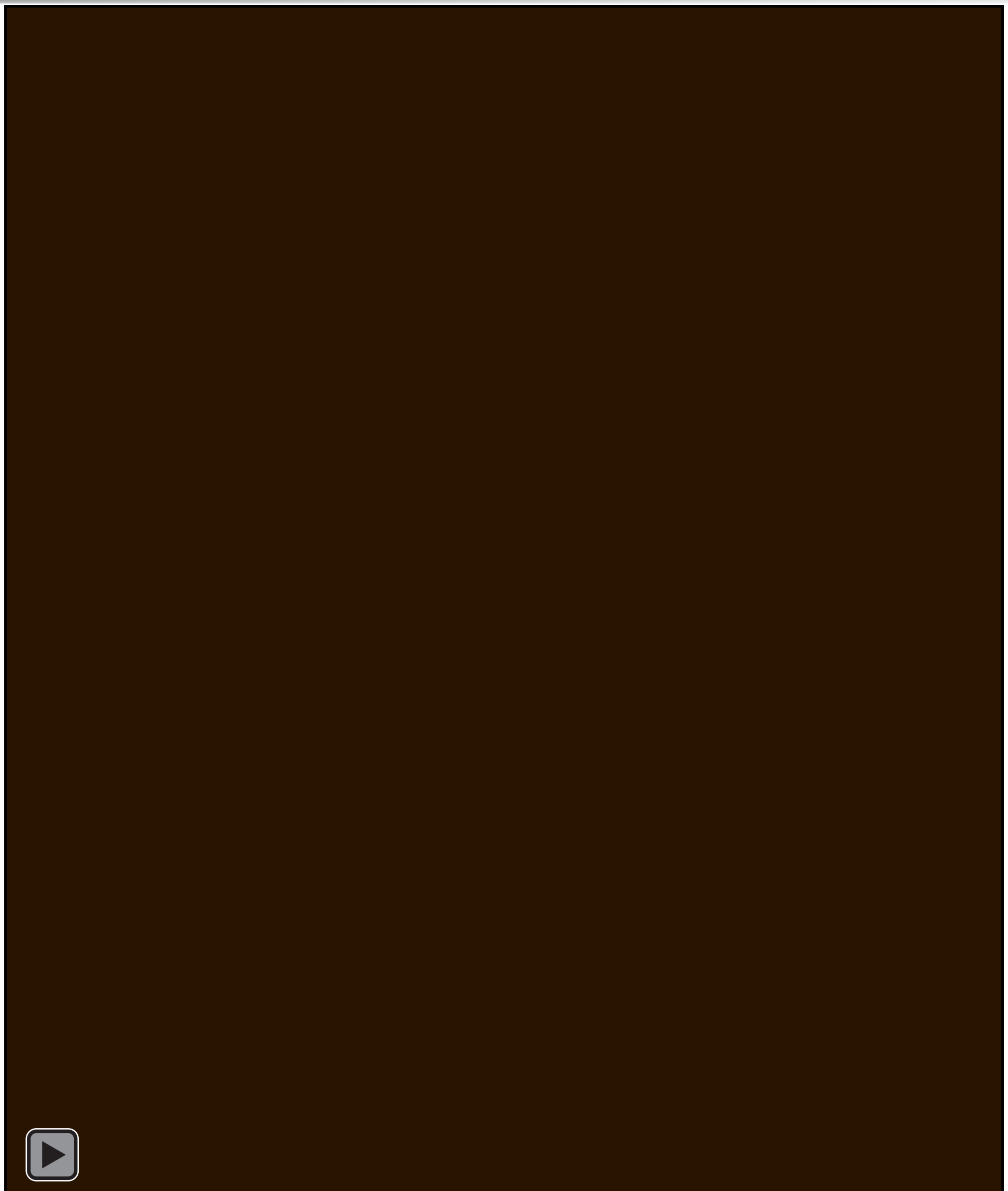
# ④
# Demonstration Analysis
# in 3D

# Tension of 3D Cube



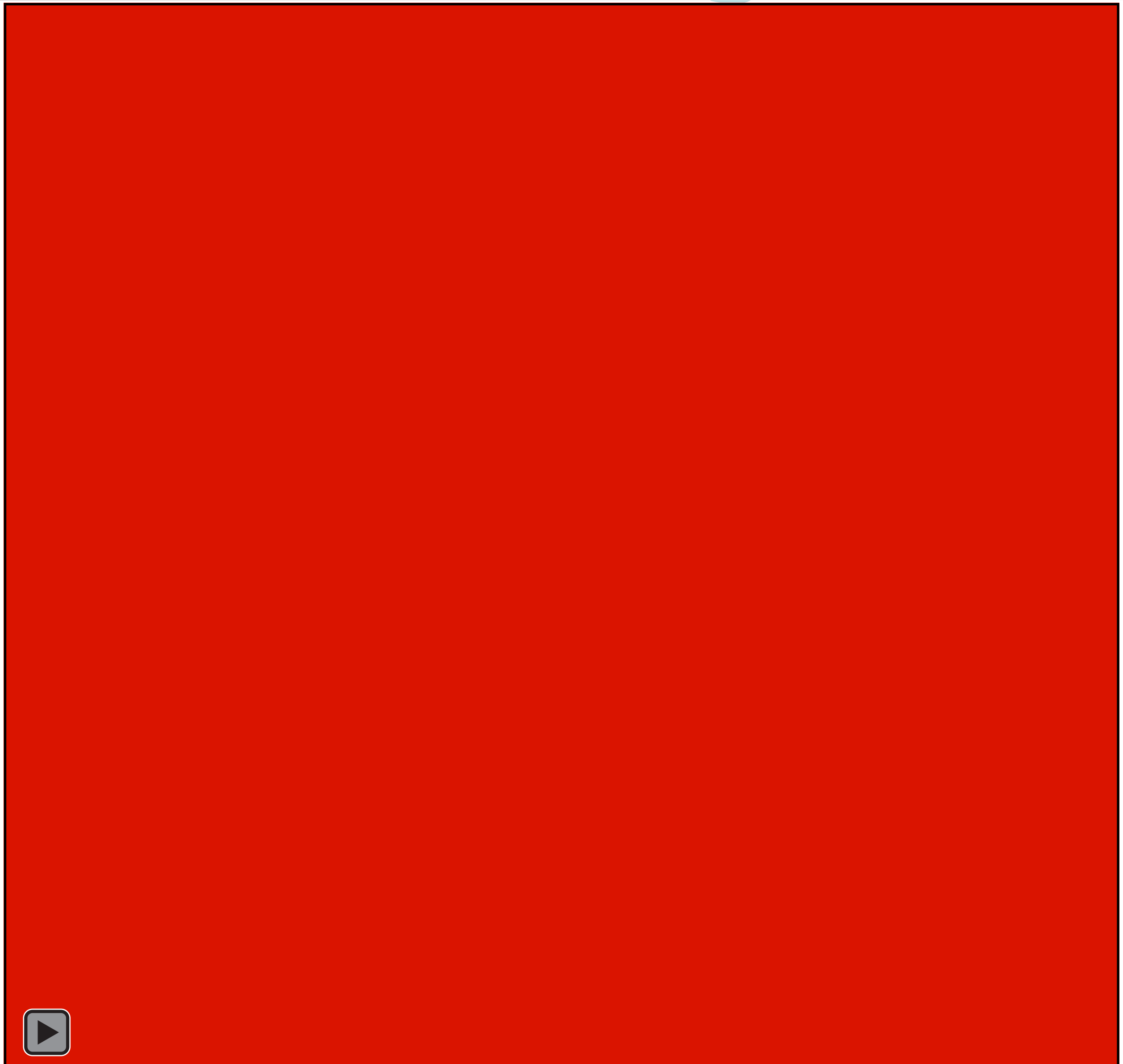
- Static, 3D
- 1/8 model of a cube
- Neo-Hookean hyperelastic body
- All 1st order tetrahedral elements
- Global rezoning every 10 timesteps
- Up to 300% nominal strain

# Twist of 3D Cuboid



- Static, 3D
- 1 x 2 x 4 m size
- Henkey's elastic body of $\nu = 0.45$
- All 1st order tetrahedral elements
- Global rezoning every 30 degree
- Up to 360 degree rotation

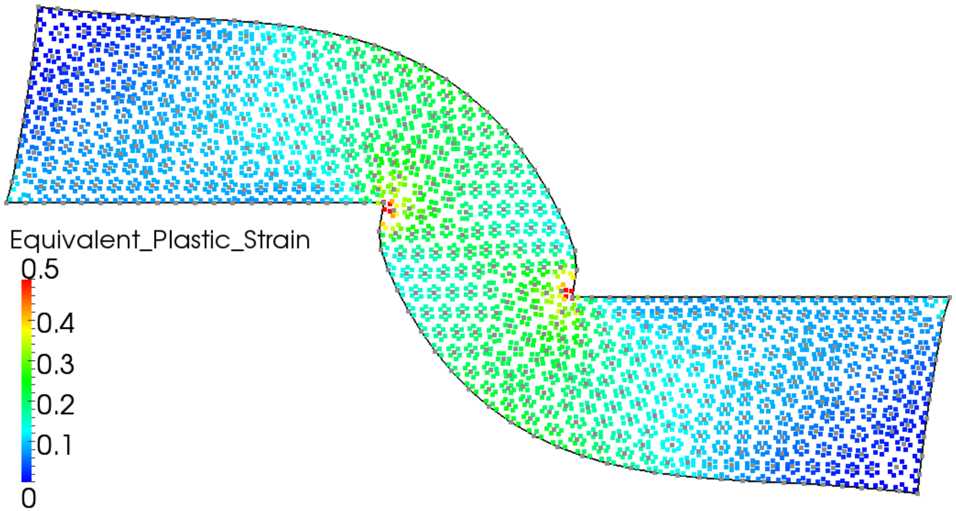
# Punching of 3D Cuboid

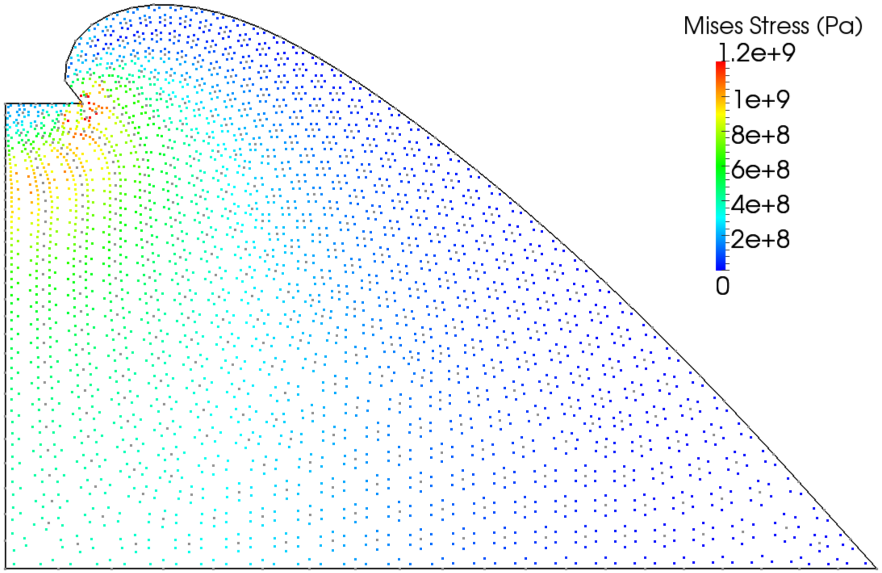


- Static, 3D
- 1/4 model
- 2 x 3 x 4 m size
- Henkey's elastic body of $\nu = 0.2$
- All 1st order tetrahedral elements
- Global rezoning every 10 timesteps
- radius of punch $R = 1$ m
- punch up to 1/3 height

# Take-Home Messages

1. Our method is as **stable** as the *explicit* method and as **accurate** as the *implicit* method.
2. The *Implicit* IEE is useful not only for FE rezoning but also Meshfree/Particle methods.




See the e-book of PARTICLES2011
or our full-paper in Int. J. Numer. Meth. Engng (2012)
in detail.

# Summary and Future Work

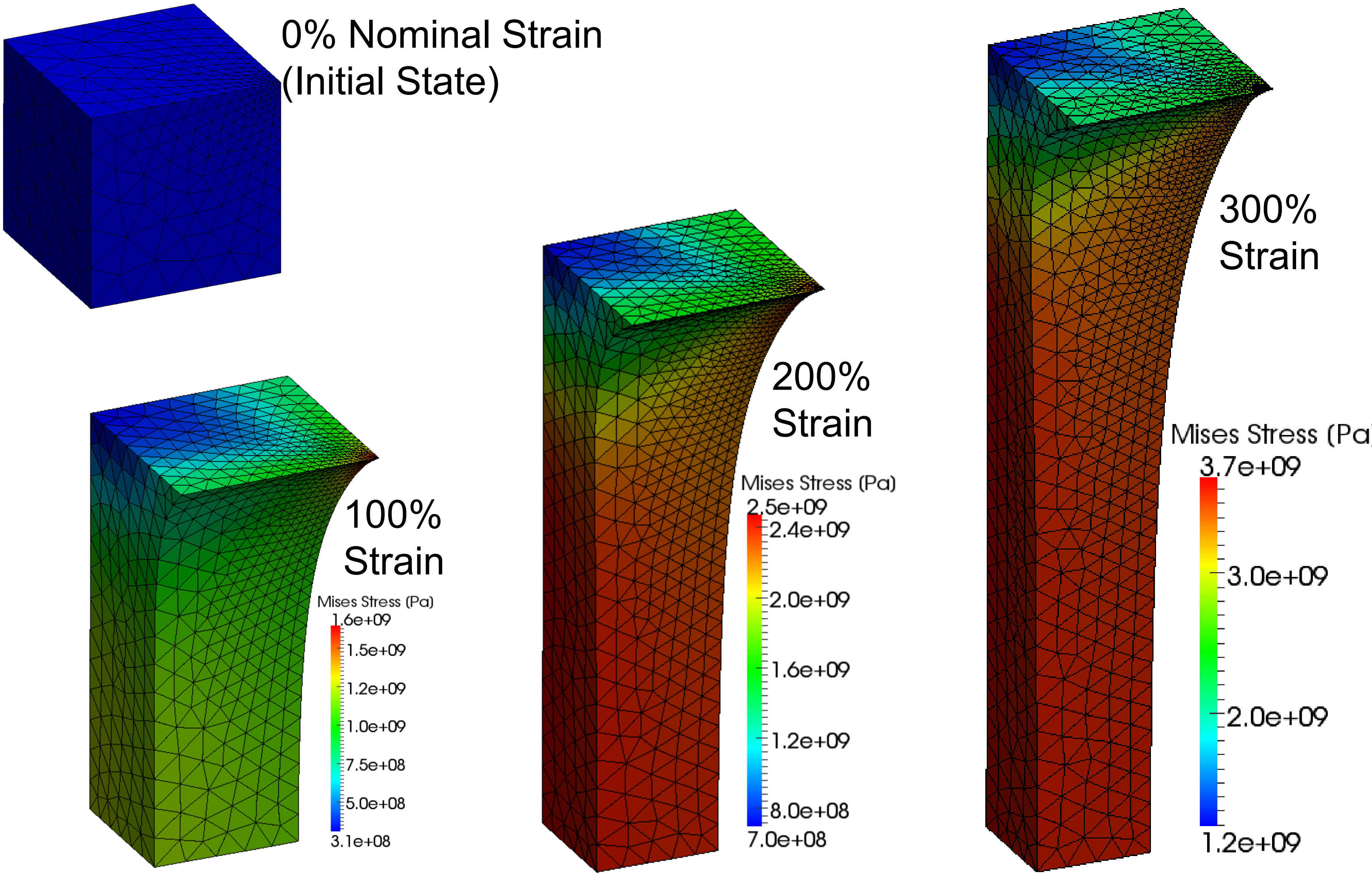
- Summary
  - A new ***implicit*** FE rezoning method for severely large deformation analysis is proposed.
  - It solves the IEE instead of the standard EE.
  - It maps $f^{\mathrm{ext}}$ in addition to the other states.
  - **Its accuracy and stability** are verified.

- Future Work
  - More V&V
  - SFEM implementation
  - Apply to contact forming, crack propagation, etc.

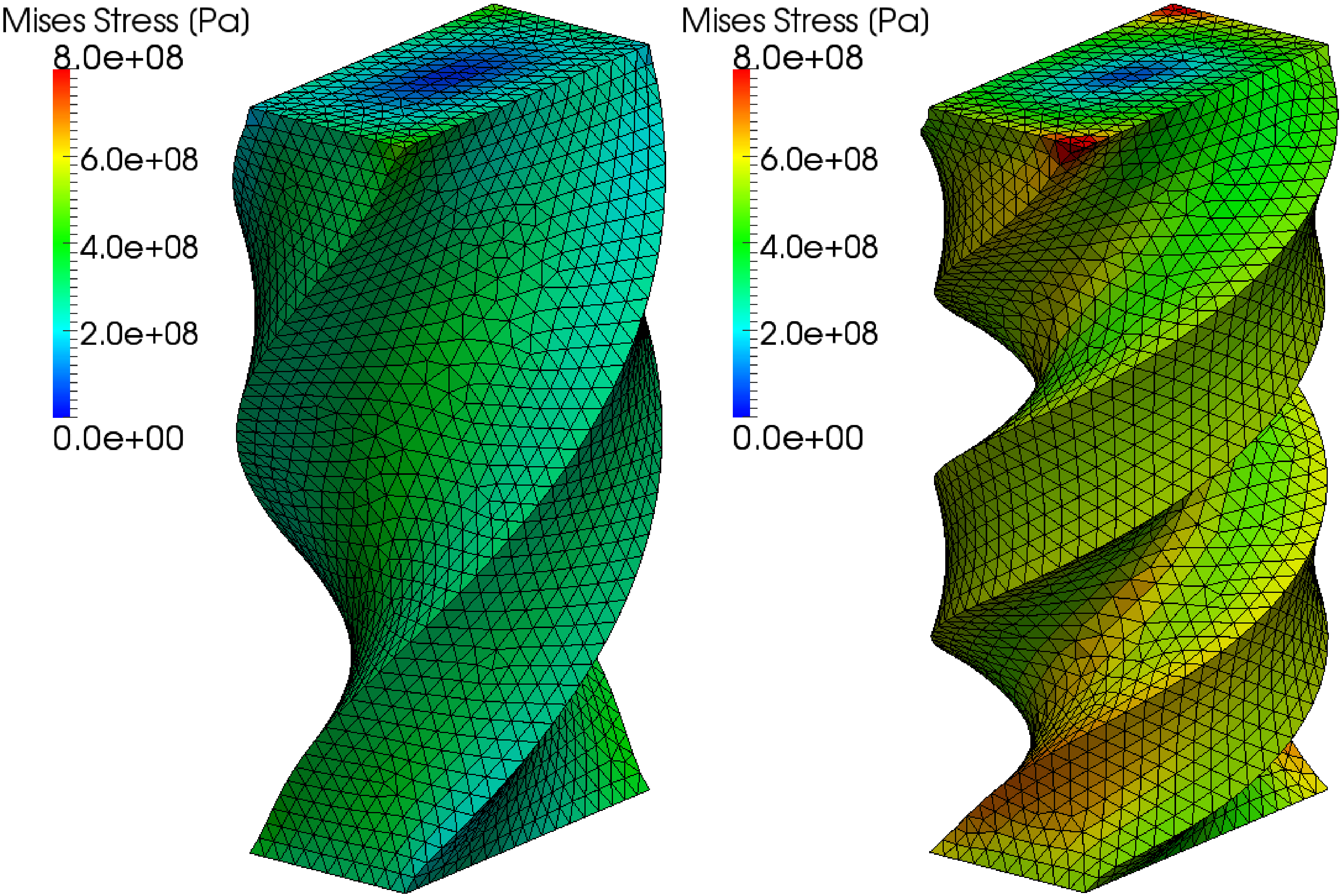
Thank you for your kind attention!

# Appendix

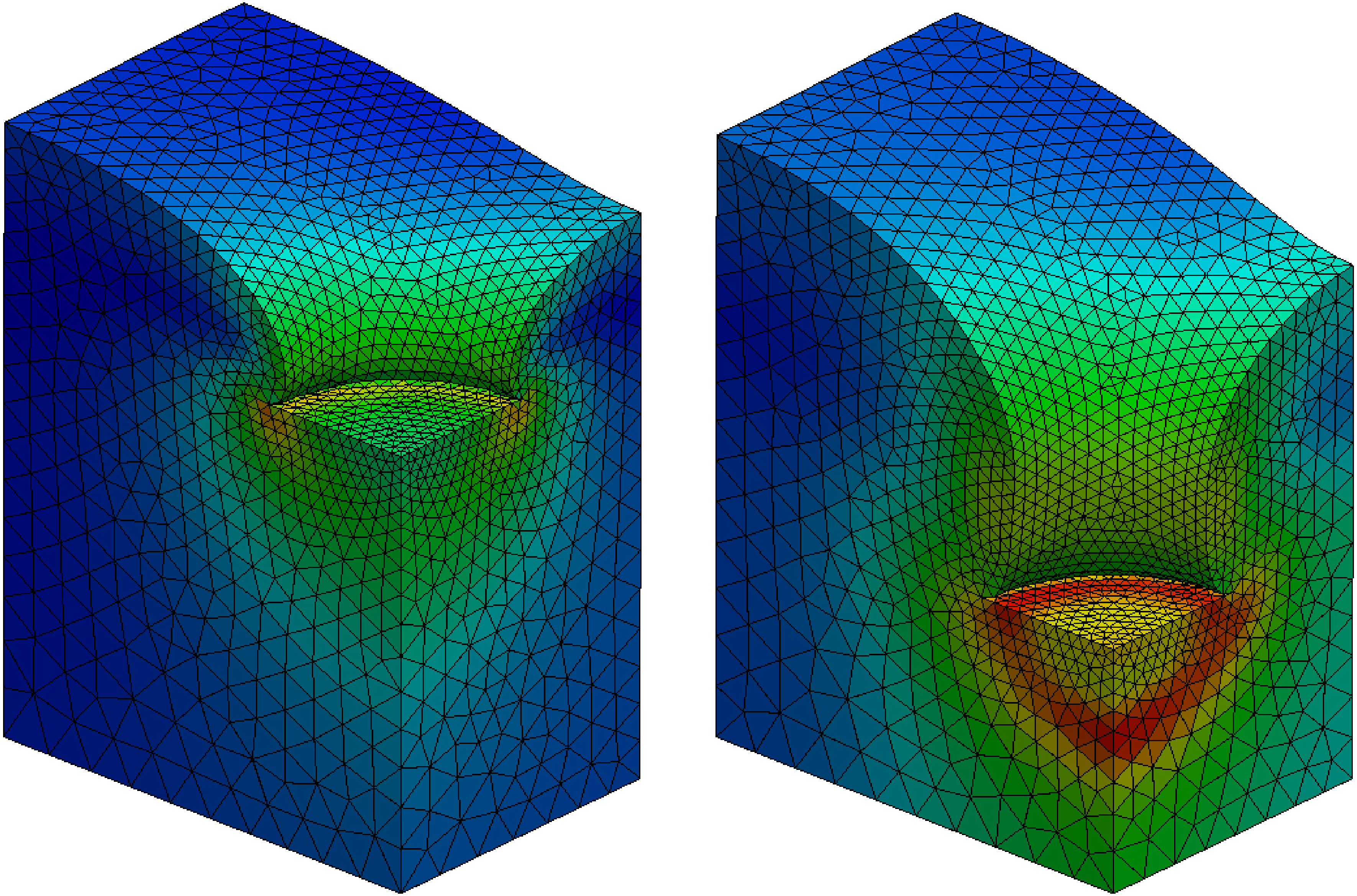
# Tension of 3D Cube



# Twist of 3D Cuboid



# Punching of 3D Cuboid



# Mapping of $f^{\text{ext}}$

**Boil down to the following minimization problem:**

- Unknown

  nodal $f^{\text{ext}}$ on the new mesh surface

- Cost Function

  ∑ || {surface traction on the new mesh face}

  – {surface traction on the old mesh face} ||$^2$

- Constraints
  - ∑ {new nodal $f^{\text{ext}}$} $=$ ∑ {old nodal $f^{\text{ext}}$}
  - ∑ {new nodal $x \times f^{\text{ext}}$} $=$ ∑ {old nodal $x \times f^{\text{ext}}$}

Solve it with Lagrange multiplier method

# Derivation of Stiffness Matrix (1/2)

- Relation between $\dot{\Pi}_t$ and $\dot{T}$:

$$\dot{\boldsymbol{\Pi}}_t \equiv \dot{\boldsymbol{T}} + \mathrm{tr}(\boldsymbol{L})\boldsymbol{T} - \boldsymbol{L}\boldsymbol{T}$$

- Relation between $\dot{\square}$ and Jaumann rate:

$$\dot{\boldsymbol{T}} \equiv \mathring{\boldsymbol{T}} + \boldsymbol{W}\boldsymbol{T} - \boldsymbol{T}\boldsymbol{W}$$

- Erasing $\dot{T}$:

$$\dot{\boldsymbol{\Pi}}_t^T = \mathring{\boldsymbol{T}} + \boldsymbol{W}\boldsymbol{T} - \boldsymbol{T}\boldsymbol{W} + \mathrm{tr}(\boldsymbol{L})\boldsymbol{T} - \boldsymbol{T}\boldsymbol{L}^T$$

- Constitutive equation (e.g. Hencky's):

$$\mathring{\boldsymbol{T}} = \boldsymbol{C}_{\mathrm{L}} : \boldsymbol{D}$$

- Erasing $\mathring{\boldsymbol{T}}$:

$$\dot{\boldsymbol{\Pi}}_t^T = \boldsymbol{C}_{\mathrm{L}} : \boldsymbol{D} + \boldsymbol{W}\boldsymbol{T} - \boldsymbol{T}\boldsymbol{W} + \mathrm{tr}(\boldsymbol{L})\boldsymbol{T} - \boldsymbol{T}\boldsymbol{L}^T$$

# Derivation of Stiffness Matrix (2/2)

- Rewrite in matrix form:

$$\{\dot{\Pi}_t^T\} = [C_\mathrm{L}]\{D\} + [C_\mathrm{N}]\{L\}$$

where

$$[C_\mathrm{N}] = \begin{bmatrix}
0 & T_{xx} & T_{xx} & 0 & 0 & -T_{xy} & 0 & -T_{zx} & 0 \\
T_{yy} & 0 & T_{yy} & -T_{xy} & 0 & 0 & 0 & 0 & -T_{yz} \\
T_{zz} & T_{zz} & 0 & 0 & -T_{zx} & 0 & -T_{yz} & 0 & 0 \\
T_{xy} & 0 & T_{xy} & \frac{T_{yy}-T_{xx}}{2} & \frac{T_{yz}}{2} & \frac{-T_{yy}-T_{xx}}{2} & \frac{-T_{zx}}{2} & \frac{-T_{yz}}{2} & \frac{-T_{zx}}{2} \\
T_{zx} & T_{zx} & 0 & \frac{T_{yz}}{2} & \frac{T_{zz}-T_{xx}}{2} & \frac{-T_{yz}}{2} & \frac{-T_{xy}}{2} & \frac{-T_{zz}-T_{xx}}{2} & \frac{-T_{xy}}{2} \\
0 & T_{xy} & T_{xy} & \frac{-T_{yy}-T_{xx}}{2} & \frac{-T_{yz}}{2} & \frac{-T_{yy}+T_{xx}}{2} & \frac{T_{zx}}{2} & \frac{-T_{yz}}{2} & \frac{-T_{zx}}{2} \\
T_{yz} & T_{yz} & 0 & \frac{-T_{zx}}{2} & \frac{-T_{xy}}{2} & \frac{T_{zx}}{2} & \frac{T_{zz}-T_{yy}}{2} & \frac{-T_{xy}}{2} & \frac{-T_{zz}-T_{yy}}{2} \\
0 & T_{zx} & T_{zx} & \frac{-T_{yz}}{2} & \frac{-T_{zz}-T_{xx}}{2} & \frac{-T_{yz}}{2} & \frac{-T_{xy}}{2} & \frac{-T_{zz}+T_{xx}}{2} & \frac{T_{xy}}{2} \\
T_{yz} & 0 & T_{yz} & \frac{-T_{zx}}{2} & \frac{-T_{xy}}{2} & \frac{-T_{zx}}{2} & \frac{-T_{zz}-T_{yy}}{2} & \frac{T_{xy}}{2} & \frac{-T_{zz}+T_{yy}}{2}
\end{bmatrix}$$

- Stiffness Matrix

$$[K^+] = \sum_{e \in \mathbb{E}} \int_{\Omega_e^+} [B_\mathrm{L}^+]^T [C_\mathrm{L}][B_\mathrm{L}^+] + [B_\mathrm{N}^+]^T [C_\mathrm{N}][B_\mathrm{N}^+] \, \mathrm{d}\Omega$$

# Characteristics of Meshfree Method

## Advantage

- Locking free
- Easy to add/remove nodes

## Disadvantage

- High cost to construct a stable shape function
- High cost to correct [B] to satisfy the divergence-free condition
- High cost to treat multiple materials, concave boundaries, contact, etc.

# Characteristics of Static-Explicit Method

## Advantage

- No convergence calculation
- Solution can stably be obtained
  (even if there exist no static solution)

## Disadvantage

- Equilibrium is not guaranteed due to error accumulation
- Necessitates small $\Delta t$ with r-min method in plasticity

# Comparison of IEE to Stiffness Eq.

[Standard Static-*Explicit* EE]

NOT equilibrium equation BUT stiffness equation without trial values

$$[K]\{\Delta u\} = \{\Delta f^{\text{ext}}\}$$

$$\left(= \{f^{\text{ext}}(t+\Delta t)\} - \{f^{\text{int}}(t)\}\right)$$

[*Implicit* IEE]

$$\{\Delta f^{\text{ext}}(\boldsymbol{u}^{+})\} - \{\Delta f^{\text{int}}(\boldsymbol{u}^{+})\} = \{0\},$$

$$\{\Delta f^{\text{ext}}\} = \sum_{s\in\mathbb{S}} \int_{\Gamma_s^{+}} [N^{+}]^T \{\Delta \underline{t}_t\}\ \mathrm{d}\Gamma + \sum_{e\in\mathbb{E}} \int_{\Omega_e^{+}} \rho^{+} [N^{+}]^T \{\Delta g\}\ \mathrm{d}\Omega,$$

$$\{\Delta f^{\text{int}}\} = \sum_{e\in\mathbb{E}} \int_{\Omega_e^{+}} [B_{\mathrm{N}}^{+}]^T \{\Delta \Pi_t^T\}\ \mathrm{d}\Omega,$$